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Transient heat conduction in one-dimensional composite slab. A 'natural' analytic approach

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Abstract

The transient response of one-dimensional multilayered composite conducting slabs to sudden variations of the temperature of the surrounding fluid is analysed. The solution is obtained applying the method of separation of variables to the heat conduction partial differential equation. In separating the variables, the thermal diffusivity is retained on the side of the modified heat conduction equation where the time-dependent function is collected. This choice is the *essence* of composite medium analysis itself. In fact, it 'naturally' gives the relationship between the eigenvalues for the different regions and then yields a transcendental equation for the determination of the eigenvalues in a less complex form than the ones resulting from the application of traditional techniques. A new type of orthogonality relationship is developed by the author and used to obtain the final complete series solution. The errors, which develop when the higher terms in the series solution are neglected, are also investigated. Some calculated results of a numerical example are shown in a graphical form, by using dimensionless groups, and therefore discussed.

Keywords: Composites; Conduction; Transient

1. Introduction

Numerous applications in engineering require a detailed knowledge of the transient temperature distribution and heat flux within slabs composed of two or more different layers.

One area is that of a new type of double heat flux conductimeter [1]. It is of interest when the transient thermal response of the apparatus (i.e., two fluxmeter plates and the sample material between them) has to be evaluated in order to establish the time required to reach its steady conditions. A second area of application is in the field of thermal analyses of buildings [2]. In fact, for cooling and heating load calculations, it is a basic stage to describe with great accuracy the non steady-state thermal behaviour of multi-layer walls and rooms.

The analysis of one-dimensional transient heat conduction in a composite slab consisting of several different layers in contact may be performed following different analytical approaches.

1. The *orthogonal expansion technique* [3–6]. It is particularly suitable for solving transient problems of composite medium of finite thickness without internal heat generation (exact closed-form solution). This technique was developed first by Vodicka [3].
2. The *quasi-orthogonal expansion technique* [7,8]. It solves the same composite domain problems listed

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Nomenclature

Bi	Biot number based on first layer: hL_1/k_1
c	integration coefficient
$f(x)$	arbitrary initial temperature distribution for the slab
$F(x)$	arbitrary initial temperature difference: $T_\infty - f(x)$
h	combined heat transfer coefficient (convection and long-wave thermal radiation)
k	thermal conductivity
L	thickness
N	norm
Q	heat exchanged per unit of area
t	time
T	slab temperature
T_0	uniform initial temperature for the plane wall
T_∞	fluid temperature
x	space coordinate
X'	eigenfunction

Greek symbols

α	thermal diffusivity
β	dimensionless eigenvalue based on first layer: λL_1
γ	geometric ratio: L_2/L_1
δ	thermal diffusivity ratio: α_2/α_1
ϵ	dimensionless error
θ	temperature difference: $T_\infty - T$
θ_0	uniform initial temperature difference: $T_\infty - T_0$
Θ	dimensionless temperature: θ/θ_0
κ	thermal conductivity ratio: k_2/k_1
λ	eigenvalue (constant of separation)
ξ	dimensionless space coordinate: x/L_1
Π	function defined in Appendix A
τ	dimensionless time (or Fourier number) based on first layer: $\alpha_1 t/L_1^2$
Φ	dimensionless heat exchanged, $Q/Q_{\max}$, where $Q_{\max}$ is defined by Eq. (43)
Ψ	approximate dimensionless temperature: $\Theta - \epsilon$

Subscripts

i	index (i.e., 1 or 2)
m	integer number (positive)
max	maximum
p	number of eigenvalues used in the solution
1	first layer ($-L_1 \le x \le 0$), left side of the composite plate
2	second layer ($0 \le x \le L_2$), right side of the composite plate

Superscripts

+	dimensionless

before (exact closed-form solution). This technique was independently developed first by Tittle [7].

3. The *Laplace transform method* [9]. This method is convenient for the solution of unsteady problems of composite medium involving regions of infinite and semiinfinite thickness (exact closed-form solution).
4. The *Green's function approach* [9–12]. Transient problems on conduction of heat in composite solids with energy generation are usually best solved by this approach (exact closed-form solution).
5. The *Galerkin procedure* [13–15]. This procedure may be used for solving accurately several transient conduction problems in composite geometries. Although it gives an approximate closed-form solution (quite accurate), it offers the benefits and limitations one expects from an exact solution.
6. The *finite integral transform* technique [16]. It allows the unsteady problem for a multi-region medium with time-dependent heat transfer coefficient to be solved (approximate closed-form solution).

In problems involving two or three space-variables, a great economy in notation can be achieved by making *integral transforms* on the space-variables [9]. As an example, a Fourier integral transform was used in [17] to remove a space-variable, followed by the classical orthogonal expansion technique. Further theoretical studies on transient heat conduction in two- and three-dimensional composite solids are derived in [18–23].

Another mathematical technique which may provide a complementary point of view with respect to transient multi-layer problems, and therefore, useful physical information in spite of its algebraic and conceptual complexity, is the search for *variational principles* [24].

Therefore, there is no agreed approach to the analysis of transient composite media. Consequently there is no method of any generality for temperature calculation and no established yardstick for assessing candidate solution method. However, the analytical solution of unsteady heat conduction in a composite medium may *also* be obtained combining the efficiency of Tittle's approach [7] for the determination of the eigenvalues with the simplicity of Vodicka's approach [3] for the calculation of the corresponding orthogonal eigenfunctions, as done in this paper.

Therefore, the starting point of the analytic procedure here proposed is the application of the method of separation of variables [25] to the heat conduction equation in any one of the different regions of the solid. In separating the variables, the thermal diffusivity of each layer is retained on the side of the modified heat conduction equation (each corresponding to its own layer) where the time-dependent function is collected, like performed for the case of a single-region problem [25]. The reason for this choice is to make the solution consistent with the physical reality of the problem, since the transient response of any solid to changes in the outer boundary conditions is strictly linked to its thermal diffusivity. We just recall that the orthogonal expansion technique retains, in separating the variables, the thermal diffusivity on the side of the modified heat conduction equation where the space-dependent function is collected. Obviously, this choice does not have any physical meaning. It is merely mathematical! Similarly, the Green's function approach gives a time-variable function which is explicitly independent of thermal diffusivity.

As far as the present procedure is concerned, since the diffusivity is clearly discontinuous at the surfaces of separation of the layers, the 'natural' choice here adopted requires to fix appropriate physical constraints at these surfaces in such a way as to ensure the continuity of heat flux. In particular, such physical constraints will be expressed by means of mathematical relationships linking the eigenvalues which are usually different in the different regions of the composite medium. As a matter of fact, these relationships were found first by Mayer [26], and then independently derived by Tittle [7]. They are the *essence* of composite medium analysis itself. However, apart of my own, I know of only some transient multi-layer works [8,27] which were drawing on these sources.

Then, the application of the boundary conditions allows the determination of all unknown constants (except the one related to the initial condition), including the transcendental equation for the computation of the eigenvalues. This equation results in a less complex form than the ones obtained applying conventional techniques, according to its purely physical approach.

Once the relations between the eigenvalues of the different regions are applied, it is shown that the corresponding eigenfunctions resulting from the solution of Helmholtz equations are orthogonal satisfying a new type of orthogonality property discussed in the text, which may be defined as 'natural'. This property is similar to the one given by Vodicka [3] and allows the integration coefficients of the solution related to the initial condition to be calculated. Therefore, the exact analytical solution is given in the form of an infinite series in the space-variable with an exponential time dependency where the thermal diffusivity of the first layer explicitly appears. Then, by a suitable combination of variables and influencing quantities [11,15,21,27], this solution is represented more clearly and concisely in a dimensionless form.

A test case is presented at the end of the paper to illustrate how the proposed method works. In particular, the determination of the eigenvalues is obtained by using an efficient and accurate scheme, both graphical and numerical, discussed in the text. Since an infinite series of eigenvalues cannot be taken into account, a proper number of the first eigenvalues for the complete series solution is established. This number is estimated in such a way that the exact (infinite terms in the series) and approximate (finite terms in the series) temperatures differ by not more than 4% in the heaviest conditions (i.e., initial time and boundary surfaces), which is acceptable in most applications of engineering. The estimated number can notably be reduced for large times, but it is here considered constant in view of the high computational technology available today. Some research work on this subject was presented and published in unsteady heat conduction literature, but only for transient single-layer problem [28]. Finally, temperature profiles within the two-layer composite slab and heat exchanged during the transient heat transfer between slab and surrounding fluid are presented graphically by means of non-dimensional parameters.

2. Governing equations

Consider a composite plane wall consisting of two parallel layers as shown in Fig. 1. Let k_1 and k_2 be thermal conductivities, and α_1 and α_2 the thermal diffusivities for the first and second layers, respectively. Initially ($t = 0$) the two-region plate of finite thickness, which is confined to the domain $-L_1 \leq x \leq L_2$, is at a

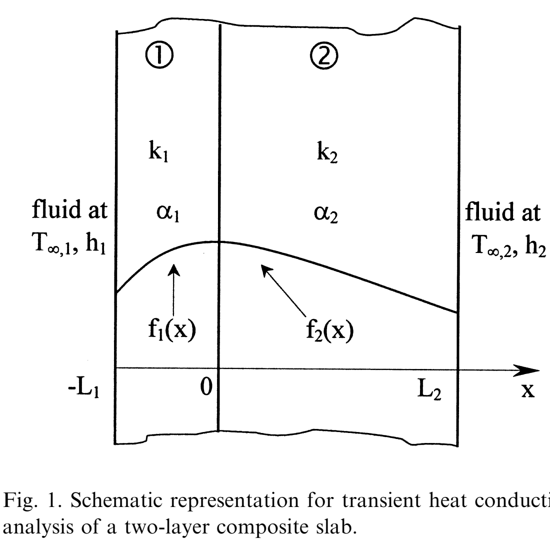

Fig. 1. Schematic representation for transient heat conduction analysis of a two-layer composite slab.

specified temperature $f(x)$. Suddenly, at $t = 0$, both boundary surfaces of the composite slab are subjected to combined convection and long-wave thermal radiation heat flux.

The assumptions made in deriving the mathematical formulation of this time-dependent heat conduction problem are:

1. there is no energy generation within the plane-parallel solid;
2. the thermal conductivity and the thermal diffusivity are temperature independent and uniform within each layer;
3. the two-layer wall is sufficiently large in the y and z directions in comparison to its thickness in the x direction;
4. the fluid temperatures $T_{\infty,1}$ and $T_{\infty,2}$ are maintained spatially uniform;
5. the combined heat transfer coefficients h_1 and h_2 are constant in correspondence to each boundary surface;
6. the fluid temperatures are maintained constant for times $t > 0$ and are equal in values, i.e. $T_{\infty,1} = T_{\infty,2} = T_{\infty}$.

Because of third, fourth and fifth assumptions, the heat conduction problem herein under discussion may be considered one-dimensional. Therefore, its final mathematical modelling in a rectangular coordinate system is given as $(t \geq 0)$

$$\frac{\partial^2 \theta_i}{\partial x^2} = \frac{1}{\alpha_i} \frac{\partial \theta_i}{\partial t} \tag{1}$$

$$\mp k_i \left(\frac{\partial \theta_i}{\partial x} \right)_{x = \mp L_i} + h_i \theta_i(x = \mp L_i, t) = 0 \tag{2}$$

$$\theta_i(x, t = 0) = F_i(x) \tag{3}$$

where the negative sign is valid when $i = 1$; conversely, when $i = 2$. The determination of the temperature distributions θ_i as functions of both time t and position x requires to assign two further boundary conditions, which are related to the surface of separation $x = 0$ of the two layers. They are given for perfect thermal contact [28] as $(t > 0)$

$$\theta_1(x = 0, t) = \theta_2(x = 0, t) \tag{4}$$

$$k_1 \left(\frac{\partial \theta_1}{\partial x} \right)_{x=0} = k_2 \left(\frac{\partial \theta_2}{\partial x} \right)_{x=0} \tag{5}$$

Fig. 1 shows that the origin $x = 0$ adopted for the frame of reference coincides with the plane of separation of the two regions. This choice allows the analytical treatment to be notably simplified when both the inner and outer boundary conditions are applied. However, it does not allow composites of cylindrical and spherical layers to be treated.

3. Solution of governing equations

Eq. (1) can be solved by using the method of separation of variables [25]. As it is well-known, the dependent variables θ_i may be separated in the form:

$$\theta_i(x, t) = X_i(x) \cdot G_i(t) \tag{6}$$

When Eq. (6) is introduced into Eq. (1), we have:

$$\frac{1}{X_i(x)} \frac{\mathrm{d}^2 X_i(x)}{\mathrm{d}x^2} = \frac{1}{\alpha_i G_i(t)} \frac{\mathrm{d}G_i(t)}{\mathrm{d}t} = -\lambda_i^2 \tag{7}$$

where the constants λ_i are the so-called *separation constants*, each corresponding to its own layer. It may be noted that, in separating the variables, the thermal diffusivities α_i are retained on the left-hand side of Eq. (7) where the time-dependent functions $G_i(t)$ are collected. This choice makes the functions $G_i(t)$ explicitly dependent on the thermal diffusivities α_i, and therefore, the analytical solution consistent with the physical reality of the problem.

The separation given by Eq. (7) yields the following ordinary differential equations for the determination of the unknown functions $(t \geq 0)$:

$$\frac{\mathrm{d}G_i(t)}{\mathrm{d}t} + \lambda_i^2 \alpha_i G_i(t) = 0 \tag{8}$$

$$\frac{\mathrm{d}^2 X_i(x)}{\mathrm{d}x^2} + \lambda_i^2 X_i(x) = 0 \tag{9}$$

The solutions for the time-variable functions $G_i(t)$ are immediately obtained from Eq. (8) as $(t \geq 0)$

$$G_i(t) = \mathrm{e}^{-\lambda_i^2 \alpha_i t} \tag{10}$$

Instead, the solutions for the space-dependent functions $X_i(x)$ are obtained by solving *Helmholtz equation* (9). They are taken as

$$X_i(x) = a_i \cos(\lambda_i x) + b_i \sin(\lambda_i x) \tag{11}$$

where a_i and b_i are the *integration constants* related to the first and second layers. Therefore, the functions $\theta_i(x, t)$ are readily determined.

3.1. Application of boundary conditions

By requiring that the solutions $\theta_i(x, t)$ satisfy the boundary conditions (2), (4) and (5), the following set of algebraic equations is obtained:

$$a_1 = b_1 \Pi_1(\lambda_1) \tag{12}$$

$$a_2 = -b_2 \Pi_2(\lambda_2) \tag{13}$$

$$a_1 \mathrm{e}^{-\lambda_1^2 \alpha_1 t} = a_2 \mathrm{e}^{-\lambda_2^2 \alpha_2 t} \tag{14}$$

$$k_1 b_1 \lambda_1 \mathrm{e}^{-\lambda_1^2 \alpha_1 t} = k_2 b_2 \lambda_2 \mathrm{e}^{-\lambda_2^2 \alpha_2 t} \tag{15}$$

where the functions $\Pi_1(\lambda_1)$ and $\Pi_2(\lambda_2)$ are given in Appendix A. Since the thermal diffusivity is discontinuous at the interface $x = 0$, Eqs. (14) and (15) are *only* verified when:

$$a_1 = a_2 \tag{16}$$

$$\frac{\lambda_2}{\lambda_1} = \pm\sqrt{\frac{\alpha_1}{\alpha_2}} \tag{17}$$

$$b_2 = \pm b_1 \left(\frac{k_1}{k_2}\right)\sqrt{\frac{\alpha_2}{\alpha_1}} \tag{18}$$

Eq. (17) relates the separation constants λ_1 and λ_2 for the first and second layers, as was found in [7,26]. Therefore, Eqs. (12), (13), (16), (17) and (18) represent a system of five nonlinear simultaneous algebraic equations in six unknown variables, i.e. a_1, b_1, λ_1 and a_2, b_2, λ_2 (the initial condition has still to be applied!).

Bearing in mind Eqs. (12), (13), (17) and (18) and setting $\lambda_1 = \lambda$ and $b_1 = c$, the functions $G_i(t)$ and $X_i(x)$ given by Eqs. (10) and (11) may be rewritten as

$$G_1(t) = \mathrm{e}^{-\lambda^2 \alpha_1 t} \quad (t \geq 0) \tag{19}$$

$$G_2(t) = \mathrm{e}^{-\lambda^2 \alpha_1 t} \quad (t \geq 0) \tag{20}$$

$$X_1(x) = c\big[\sin(\lambda x) + \Pi_1(\lambda)\cos(\lambda x)\big] \quad (-L_1 \leq x \leq 0) \tag{21}$$

$$X_2(x) = c\left(\frac{k_1}{k_2}\right)\sqrt{\frac{\alpha_2}{\alpha_1}}\Big[\sin\Big(\sqrt{\alpha_1/\alpha_2}\lambda x\Big) - \Pi_2'(\lambda)\cos\Big(\sqrt{\alpha_1/\alpha_2}\lambda x\Big)\Big] \quad (0 \leq x \leq L_2) \tag{22}$$

where the functions $\Pi_1(\lambda)$ and $\Pi_2'(\lambda)$ are still shown in Appendix A. Therefore, the solutions $\theta_1(x, t)$ and $\theta_2(x, t)$ have the following expressions:

$$\theta_1(x, t) = c\big[\sin(\lambda x) + \Pi_1(\lambda)\cos(\lambda x)\big]\mathrm{e}^{-\lambda^2 \alpha_1 t} \quad (-L_1 \leq x \leq 0;\ t \geq 0) \tag{23}$$

$$\theta_2(x, t) = c\left(\frac{k_1}{k_2}\right)\sqrt{\frac{\alpha_2}{\alpha_1}}\Big[\sin\Big(\sqrt{\alpha_1/\alpha_2}\lambda x\Big) - \Pi_2'(\lambda)\cos\Big(\sqrt{\alpha_1/\alpha_2}\lambda x\Big)\Big]\mathrm{e}^{-\lambda^2 \alpha_1 t} \quad (0 \leq x \leq L_2;\ t \geq 0) \tag{24}$$

Moreover, bearing in mind again Eqs. (12), (13), (17) and (18) and setting $\lambda_1 = \lambda$, the algebraic equation (16) may be rewritten in the following form:

$$\Pi_1(\lambda) + \left(\frac{k_1}{k_2}\right)\sqrt{\frac{\alpha_2}{\alpha_1}}\Pi_2'(\lambda) = 0 \tag{25}$$

Hence Eqs. (23) and (24) simultaneously satisfy Eqs. (1), (2), (4) and (5), where c is an arbitrary constant and λ is any root other than zero of the transcendental equation (25). It may be proven (Section 5) that its roots (called *eigenvalues*) are infinite, distinct and real: $\lambda_1 < \lambda_2 < \ldots \lambda_m < \ldots \quad (m = 1,\ 2,\ 3,\ \ldots)$. Therefore, there are numerous solutions having the forms (23) and (24), each corresponding to a consecutive value of the eigenvalues λ_m:

$$\theta_{1,m}(x, t) = c_m X'_{1,m}(x)\mathrm{e}^{-\lambda_m^2 \alpha_1 t} \quad (-L_1 \leq x \leq 0;\ t \geq 0) \tag{26}$$

$$\theta_{2,m}(x, t) = c_m\left(\frac{k_1}{k_2}\right)\sqrt{\frac{\alpha_2}{\alpha_1}}X'_{2,m}(x)\mathrm{e}^{-\lambda_m^2 \alpha_1 t} \quad (0 \leq x \leq L_2;\ t \geq 0) \tag{27}$$

The functions $X'_{1,m}(x)$ and $X'_{2,m}(x)$ are the eigenfunctions corresponding to the eigenvalues λ_m, and are defined as

$$X'_{1,m}(x) = \sin(\lambda_m x) + \Pi_{1,m}\cos(\lambda_m x) \quad (-L_1 \leq x \leq 0) \tag{28}$$

$$X'_{2,m}(x) = \sin\Big(\sqrt{\alpha_1/\alpha_2}\lambda_m x\Big) - \Pi'_{2,m}\cos\Big(\sqrt{\alpha_1/\alpha_2}\lambda_m x\Big) \quad (0 \leq x \leq L_2) \tag{29}$$

where $\Pi_{1,m} = \Pi_1(\lambda_m)$ and $\Pi'_{2,m} = \Pi_2'(\lambda_m)$. It may be proven that the eigenfunctions $X'_{1,m}(x)$ and $X'_{2,m}(x)$ defined before are orthogonal functions. In fact, they satisfy the new type of orthogonality relationship

$$k_2\int_{-L_1}^{0} X'_{1,\,m}X'_{1,\,n}\,\mathrm{d}x + k_1\int_{0}^{L_2} X'_{2,\,m}X'_{2,\,n}\,\mathrm{d}x = \begin{cases} 0 & \text{for } m\neq n \\ N_m & \text{for } m=n \end{cases} \tag{30}$$

which may be called 'natural' orthogonality property according to the 'natural' choice adopted in separating the variables. The constant N_m, called normalization integral (norm), is defined as

$$N_m = k_2\int_{-L_1}^{0} X'^{\,2}_{1,\,m}\,\mathrm{d}x + k_1\int_{0}^{L_2} X'^{\,2}_{2,\,m}\,\mathrm{d}x \tag{31}$$

and for this particular case it is evaluated as

$$N_m = \frac{k_2}{2}\left(1+\Pi^2_{1,\,m}\right)\left(L_1 + \frac{1}{\lambda_m^2 k_1/h_1 + h_1/k_1}\right) + \frac{k_1}{2}\left(1+\Pi'^{\,2}_{2,\,m}\right)\left[L_2 + \frac{1}{(\alpha_1/\alpha_2)\lambda_m^2 k_2/h_2 + h_2/k_2}\right] \tag{32}$$

Then the complete solution for the temperature distributions $\theta_1(x, t)$ and $\theta_2(x, t)$ is constructed by taking a linear sum of all individual solutions given by Eqs. (26) and (27) over all eigenvalues λ_m:

$$\theta_1(x, t) = \sum_{m=1}^{\infty} c_m X'_{1,\,m}(x)\mathrm{e}^{-\lambda_m^2\alpha_1 t} \quad (-L_1\le x\le 0;\ t\ge 0) \tag{33}$$

$$\theta_2(x, t) = \left(\frac{k_1}{k_2}\right)\sqrt{\frac{\alpha_2}{\alpha_1}}\sum_{m=1}^{\infty} c_m X'_{2,\,m}(x)\mathrm{e}^{-\lambda_m^2\alpha_1 t} \quad (0\le x\le L_2;\ t\ge 0) \tag{34}$$

3.2. Application of initial condition

We now constrain the solutions (33) and (34) to satisfy the initial condition (3), and obtain

$$F_1(x) = \sum_{m=1}^{\infty} c_m X'_{1,\,m}(x) \quad (-L_1\le x\le 0) \tag{35}$$

$$F_2(x) = \left(\frac{k_1}{k_2}\right)\sqrt{\frac{\alpha_2}{\alpha_1}}\sum_{m=1}^{\infty} c_m X'_{2,\,m}(x) \quad (0\le x\le L_2) \tag{36}$$

The coefficients c_m can be determined by using the orthogonality relation (30) as now described. Both sides of Eq. (35) can be multiplied by $k_2\cdot X'_{1,\,n}$, and the resulting expression is integrated with respect to x from $x=-L_1$ to $x=0$. Similarly, both sides of Eq. (36) can be multiplied by $\sqrt{\alpha_1/\alpha_2}\cdot k_2\cdot X'_{1,\,n}$, and the resulting expression is integrated from $x=0$ to $x=L_2$. We obtain

$$k_2\int_{-L_1}^{0} F_1(x)X'_{1,\,n}\,\mathrm{d}x = k_2\sum_{m=1}^{\infty} c_m\int_{-L_1}^{0} X'_{1,\,m}X'_{1,\,n}\,\mathrm{d}x \tag{37}$$

$$k_2\sqrt{\frac{\alpha_1}{\alpha_2}}\int_{0}^{L_2} F_2(x)X'_{2,\,n}\,\mathrm{d}x = k_1\sum_{m=1}^{\infty} c_m\int_{0}^{L_2} X'_{2,\,m}X'_{2,\,n}\,\mathrm{d}x \tag{38}$$

Summing up the expressions (37) and (38), and applying the orthogonality property (30), we determine the coefficients c_m as

$$c_m = \frac{k_2}{N_m}\left[\int_{-L_1}^{0} F_1(x)X'_{1,\,m}\,\mathrm{d}x + \sqrt{\frac{\alpha_1}{\alpha_2}}\int_{0}^{L_2} F_2(x)X'_{2,\,m}\,\mathrm{d}x\right] \tag{39}$$

If the composite slab is initially at a uniform temperature T_0, then $F_1(x) = F_2(x) = T_\infty - T_0 = \theta_0$ in Eq. (39), and the coefficients c_m are evaluated as

$$c_m = \frac{k_2\theta_0}{N_m\lambda_m}\left\{\left[\cos(\lambda_m L_1) - \cos\left(\sqrt{\alpha_1/\alpha_2}\lambda_m L_2\right)\right] + \left[\Pi_{1,\,m}\sin(\lambda_m L_1) - \Pi'_{2,\,m}\sin\left(\sqrt{\alpha_1/\alpha_2}\lambda_m L_2\right)\right]\right\} \tag{40}$$

3.3. Heat exchanged

The *positive* total amount of heat Q per unit of area which is exchanged between composite slab and outer fluid up to any time $t>0$ during the transient heat transfer process may be calculated as

$$Q(t) = \int_0^t \left|h_1\theta_1\left(x=-L_1, t'\right)\right|\,\mathrm{d}t' + \int_0^t \left|-h_2\theta_2\left(x=L_2, t'\right)\right|\,\mathrm{d}t' \tag{41}$$

In view of both the absolute value and distributive laws for definite integrals, Eq. (41) can be solved getting the following result:

$$Q(t)=\frac{h_1}{\alpha_1}\left|\sum_{m=1}^{\infty}\left[\frac{c_m}{\lambda_m^2}\left(1-e^{-\lambda_m^2\alpha_1 t}\right)X'_{1,m}(-L_1)\right]\right| + \frac{h_2}{\alpha_1}\left(\frac{k_1}{k_2}\right)\sqrt{\frac{\alpha_2}{\alpha_1}}\left|\sum_{m=1}^{\infty}\left[\frac{c_m}{\lambda_m^2}\left(e^{-\lambda_m^2\alpha_1 t}-1\right)X'_{2,m}(L_2)\right]\right| \quad (t>0) \tag{42}$$

where the coefficients c_m are in general given by Eq. (39). If the two-layer slab is initially at a uniform temperature T_0, then these coefficients are calculated by Eq. (40). In this case, the maximum amount of heat exchanged per unit of area, $Q_{\max}=Q(t\to\infty)$, is simply given by

$$Q_{\max}=\left(\frac{k_1}{\alpha_1}\right)L_1|\theta_0|+\left(\frac{k_2}{\alpha_2}\right)L_2|\theta_0| \tag{43}$$

4. Dimensionless temperature and heat exchanged

The theoretical solution of the considered transient heat conduction two-layer problem can be represented more clearly and concisely by the use of suitable dimensionless groups [11,15,21,27]. In fact, we have:

$$\Theta_1(\xi,\tau)=\sum_{m=1}^{\infty}c_m^+X'_{1,m}(\xi)e^{-\beta_m^2\tau} \quad (-1\le\xi\le 0;\ \tau\ge 0) \tag{44}$$

$$\Theta_2(\xi,\tau)=\frac{\sqrt{\delta}}{\kappa}\sum_{m=1}^{\infty}c_m^+X'_{2,m}(\xi)e^{-\beta_m^2\tau} \quad (0\le\xi\le\gamma;\ \tau\ge 0) \tag{45}$$

where β_m is the mth dimensionless eigenvalue, $\lambda_m L_1$. In particular, λ_m represents the mth root of the transcendental equation (25) which can be rewritten as

$$\Pi_1(\beta)=\Pi''_2(\beta) \tag{46}$$

where the functions $\Pi_1(\beta)$ and $\Pi''_2(\beta)$ are shown in Appendix A. The eigenfunctions $X'_{1,m}$ and $X'_{2,m}$, expressed by Eqs. (28) and (29), become

$$X'_{1,m}(\xi)=\sin(\beta_m\xi)+\Pi_{1,m}\cos(\beta_m\xi) \quad (-1\le\xi\le 0) \tag{47}$$

$$X'_{2,m}(\xi)=\sin\left(\beta_m\xi/\sqrt{\delta}\right)-\Pi'_{2,m}\cos\left(\beta_m\xi/\sqrt{\delta}\right) \quad (0\le\xi\le\gamma) \tag{48}$$

where $\Pi_{1,m}=\Pi_1(\beta_m)$ and $\Pi'_{2,m}=\Pi'_2(\beta_m)$. Bearing in mind Eq. (40), which is only valid when the two-region plane wall is initially at a uniform temperature θ_0, the normalized coefficients $c_m^+=c_m/\theta_0$ appearing in Eqs. (44) and (45) may be calculated as

$$c_m^+=\frac{\kappa}{N_m^+\beta_m}\left\{\cos(\beta_m)-\cos\left[(\gamma/\sqrt{\delta})\beta_m\right]+\Pi_{1,m}\sin(\beta_m)-\Pi'_{2,m}\sin\left[(\gamma/\sqrt{\delta})\beta_m\right]\right\} \tag{49}$$

where N_m^+ represents the mth dimensionless norm, defined as $N_m/(k_1L_1)$. When non-dimensional variables are used, norm N_m expressed by Eq. (32) becomes

$$N_m^+=\frac{\kappa}{2}\left(1+\Pi_{1,m}^2\right)\left(1+\frac{1}{\beta_m^2/Bi_1+Bi_1}\right)+\frac{\gamma}{2}\left(1+\Pi'^2_{2,m}\right)\left\{1+\frac{1}{\left[(\gamma/\sqrt{\delta})\beta_m\right]^2\kappa/(Bi_2\gamma)+Bi_2\gamma/\kappa}\right\} \tag{50}$$

Therefore, the dimensionless temperature distributions Θ_1 and Θ_2 depend on only the following seven dimensionless variables:

$\xi,\ \tau,\ \kappa,\ \gamma,\ Bi_1,\ Bi_2,\ \delta$

These variables are the *sole* parameters of an explicit mathematical description of all the physical processes taking place in the transient heat conduction of the considered one-dimensional two-layer slab. The number of parameters affecting the temperature distributions has been significantly reduced. It may also be observed that the dimensionless groups, τ, Bi_1 and Bi_2, are based on the first layer [21,27].

Similarly, the amount of heat $Q(t)$ per unit of area exchanged between the two-layer slab and the surrounding fluid, and expressed by Eq. (42) linked to Eq. (40), can be represented more clearly in a dimensionless form as follows

$$\Phi(\tau)=\frac{1}{1+\kappa\gamma/\delta}\left\{Bi_1\left|\sum_{m=1}^{\infty}\frac{c_m^+}{\beta_m^2}\left(1-e^{-\beta_m^2\tau}\right)X'_{1,m}(-1)\right|+\frac{Bi_2\sqrt{\delta}}{\kappa}\left|\sum_{m=1}^{\infty}\frac{c_m^+}{\beta_m^2}\left(e^{-\beta_m^2\tau}-1\right)X'_{2,m}(\gamma)\right|\right\} \quad (\tau>0) \tag{51}$$

5. Example of application

With reference to the examined transient two-region problem, we assume for the non-dimensional quantities listed before the following values: $\gamma = 2$, $\kappa = 2$, $\delta = 1$, $Bi_1 = 1$ and $Bi_2 = 2$. Of course, both the regions are initially at a uniform temperature θ_0.

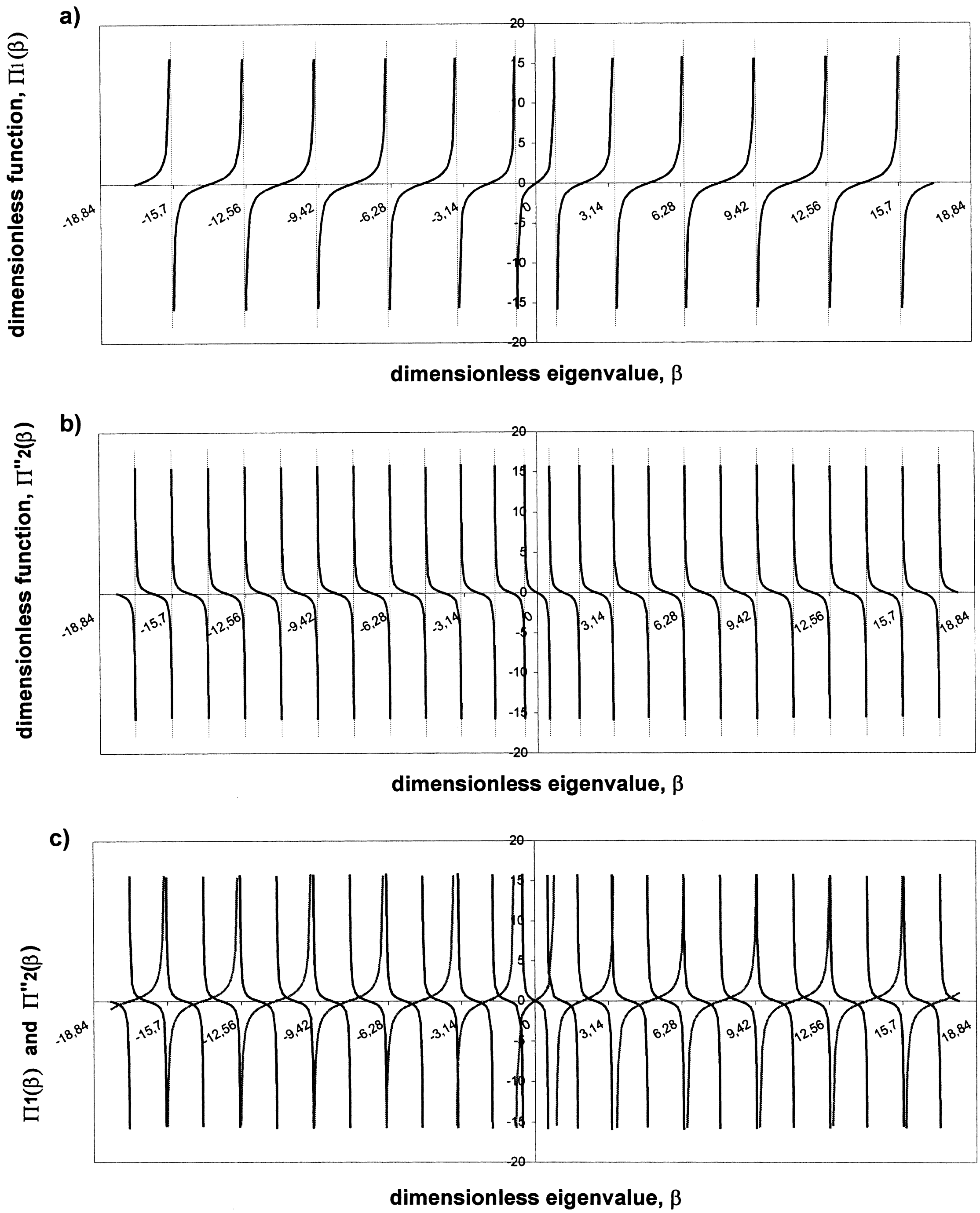

Fig. 2. Π_1 as a function of dimensionless eigenvalue β (a). Π_2'' as a function of dimensionless eigenvalue β (b). Π_1 and Π_2'' vs. β (c).

Table 1
Significant values of β for Π_1 and Π_2''

$\Pi_1(\beta)$ function		$\Pi_2''(\beta)$ function	
Vertical asymptotes ($\beta = 0$)	$\Pi_1(\beta) = 0$	Vertical asymptotes ($\beta = 0$)	$\Pi_2''(\beta) = 0$
±0.8603	±2.0288	±0.53845	±1.14445
±3.4256	±4.9132	±1.8218	±2.5435
±6.4373	±7.9787	±3.28915	±4.0481
±9.5293	±11.0856	±4.8148	±5.58635
±12.6453	±14.2075	±6.36115	±7.1382
±15.7713	±17.3364	±7.9168	±8.6966
–	–	±9.4773	±10.25875
–	–	±11.04075	±11.82315
–	–	±12.60595	±13.38905
–	–	±14.1724	±14.95595
–	–	±15.7397	±16.5236
–	–	±17.3076	±18.09175

5.1. Determination of the eigenvalues

The first step concerns the determination of the eigenvalues β_m. For the numerical example here proposed the functions $\Pi_1(\beta)$ and $\Pi_2''(\beta)$, formally defined in Appendix A and appearing in transcendental equation (46), become

$$\Pi_1(\beta) = \frac{\beta + \tan(\beta)}{1 - \beta\tan(\beta)}$$
$$\Pi_2''(\beta) = -\frac{1}{2}\frac{(2\beta) + 2\tan(2\beta)}{2 - (2\beta)\tan(2\beta)} \tag{52}$$

In this case, it is not possible to obtain an explicit solution for the eigenvalues β_m. However, the transcendental equation (46) linked to Eq. (52) can be solved (and the β_m values estimated) by means of both a graphical and a numerical scheme, as discussed below.

To have an idea about the distribution of the real and distinct roots of Eq. (46), it may be useful to plot the functions $\Pi_1(\beta)$ and $\Pi_2''(\beta)$ against β, as shown in Fig. 2(a) and (b), respectively.

In particular, since $\Pi_1(\beta)$ is a periodic function, its graph crosses the β-axis (where $\Pi_1 = 0$) infinite times. The corresponding values of β are the solutions to the transcendental equation $\beta + \tan(\beta) = 0$. Moreover, the curve $\Pi_1(\beta)$ has infinite vertical asymptotes, whose corresponding values of β may be determined as the solutions to the equation $1 - \beta\tan(\beta) = 0$.

Similarly, the curve $\Pi_2''(\beta)$ cuts the β-axis (where $\Pi_2'' = 0$) infinite times. The corresponding values of β represent the solutions to the equation $(2\beta) + 2\tan(2\beta) = 0$. Additionally, the curve $\Pi_2''(\beta)$ presents infinite vertical asymptotes, whose corresponding β values are given as the roots of the equation $2 - (2\beta)\tan(2\beta) = 0$.

Table 1 shows some of these significant values of β for $\Pi_1(\beta)$ and $\Pi_2''(\beta)$. If these functions are drawn on the same rectangular axes, as shown in Fig. 2(c), the β-values of the *infinite* intersection points of the two curves give the solutions (*infinite*, *distinct* and *real* roots) to the transcendental equation (46) linked to Eq. (52). At these points it will in fact result in $\Pi_1(\beta) = \Pi_2''(\beta)$.

In addition, bearing in mind Table 1, Fig. 2(c) shows that the positive roots lie one in each of the intervals listed in Table 2 (in particular, only the first 17 ranges have been considered). Once these intervals have been established, it is a simple matter 'to build' a

Table 2
First 17 roots of the transcendental equation (46) linked to Eq. (52)

m	β_m	Interval	Value of β_m
1	β_1	0.53845 → 0.8603	0.6150667
2	β_2	1.14445 → 1.8218	1.5436569
3	β_3	2.0288 → 2.5435	2.2845548
4	β_4	3.28915 → 3.4256	3.3173428
5	β_5	4.0481 → 4.8148	4.4454150
6	β_6	4.9132 → 5.58635	5.2652385
7	β_7	6.36115 → 6.4373	6.3765322
8	β_8	7.1382 → 7.9168	7.5254517
9	β_9	7.9787 → 8.6966	8.3582468
10	β_{10}	9.4773 → 9.5293	9.4877882
11	β_{11}	10.25875 → 11.04075	10.6403975
12	β_{12}	11.0856 → 11.82315	11.4771473
13	β_{13}	12.60595 → 12.6453	12.6138397
14	β_{14}	13.38905 → 14.1724	13.7672471
15	β_{15}	14.2075 → 14.95595	14.6056780
16	β_{16}	15.7397 → 15.7713	15.7460178
17	β_{17}	16.5236 → 17.3076	16.8995043

suitable 17-vector of initial guesses $\beta_{1,\ i}$, $\beta_{2,\ i}$, ..., $\beta_{17,\ i}$ for Eq. (46) and, therefore, to find its roots by standard routines. It should be noted that 'the building' of this vector in general represents the starting step and the most difficult step for reaching the convergence of any routine able to compute real roots of any real equation.

Therefore, the graphical representation of the functions $\Pi_1(\beta)$ and $\Pi_2''(\beta)$ allows the first intervals where the first real roots of Eq. (46) fall into to be established. Then, applying an algorithm based on Muller's method [29], the values of the eigenvalues can readily be obtained. They are shown in Table 2. Moreover, the negative roots are equal in absolute value to the positive ones and there are no repeated roots.

5.2. Approximation for small and large times

Once the first eigenvalues of Eq. (46) have been determined, the second step concerns the estimation of the number p of eigenvalues which has to be employed in the series of Θ_1 and Θ_2. To this aim the above series, formally represented by Eqs. (44) and (45), may also be rewritten as

$$\begin{aligned}\Theta_1(\xi,\tau) &= \sum_{m=1}^{p} c_m^{+} X'_{1,\ m}(\xi)\mathrm{e}^{-\beta_m^2\tau} + \epsilon_1(\xi,\tau,p)\\ &= \Psi_1(\xi,\tau,p) + \epsilon_1(\xi,\tau,p)\end{aligned} \qquad (53)$$

$(-1 \le \xi \le 0;\ \tau \ge 0)$

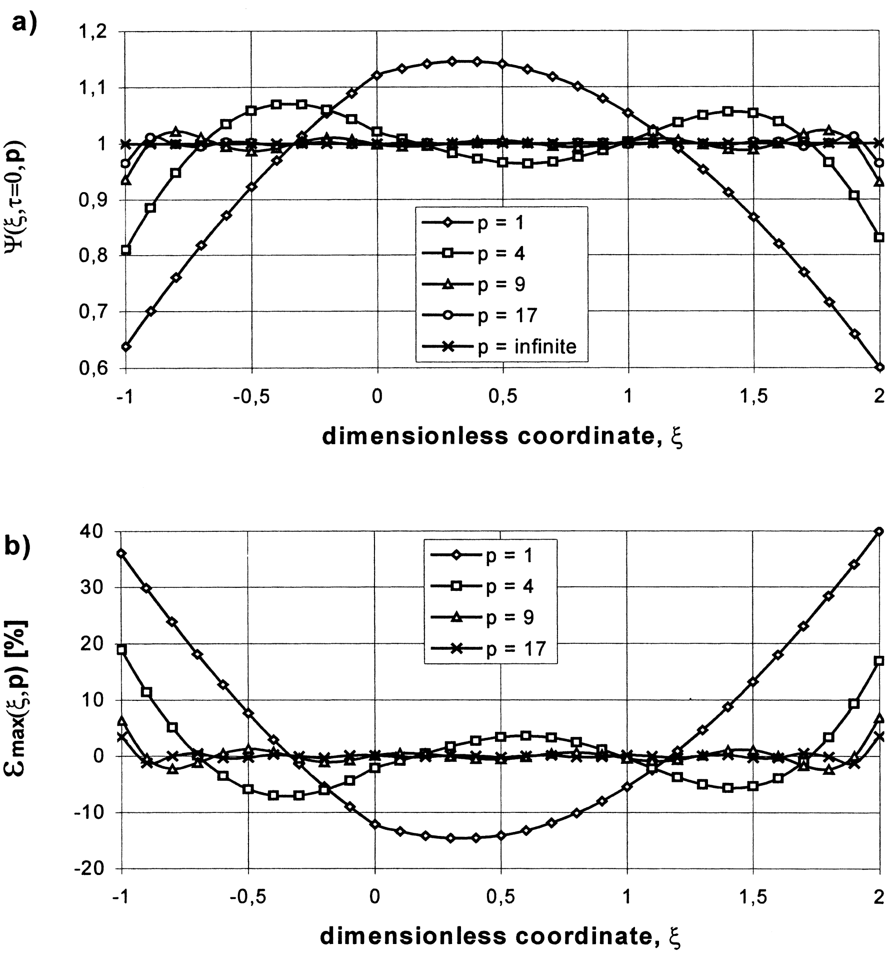

Fig. 3. Approximate dimensionless temperature Φ for $\tau = 0$ as a function of ξ with p as a parameter (a). Maximum percent error (%) vs. ξ with p as a parameter (b).

$$\Theta_2(\xi,\tau) = \frac{\sqrt{\delta}}{\kappa}\sum_{m=1}^{p} c_m^+ X'_{2,\,m}(\xi) e^{-\beta_m^2 \tau} + \epsilon_2(\xi,\tau,p) = \Psi_2(\xi,\tau,p) + \epsilon_2(\xi,\tau,p) \quad (0 \le \xi \le \gamma;\ \tau \ge 0) \tag{54}$$

where Ψ_1 and Ψ_2 represent *approximate* dimensionless temperatures depending on the eigenvalue number p. In particular, $\Psi_i(\xi, \tau, p = 1)$ $(i = 1, 2)$ may be termed the first partial sum of the series $\Theta_i(\xi, \tau)$, $\Psi_i(\xi, \tau, p = 2)$ the second partial sum, $\Psi_i(\xi, \tau, p = n)$ the nth partial sum, and so on.

Large deviations between the exact Θ_i and approximate Ψ_i solutions develop for small time τ, that is at the beginning of a heating or cooling process, since many terms of the infinite series (44) and (45) are required to calculate the temperature distribution in the composite slab. As the maximum deviation is obtained for $\tau = 0$, the time $\tau = 0$ is of interest in order to analyse the effect of the number p on the approximate solution.

Graphs of $\Psi_i(\xi, \tau = 0, p)$ for different values of p are illustrated in Fig. 3(a). This figure shows that the functions $\Psi_i(\xi, \tau = 0, p)$ are continually approximating towards a definite limit as more and more partial sums are taken (i.e., the eigenvalue number p increases), and in the limit $(p \to \infty)$ will have the sum $\Theta_i(\xi, \tau = 0) = 1$.

From Fig. 3(a) it may be noted that when $p = 17$ the difference between exact and approximate temperature is quite small. To establish some criterion under which this temperature difference can be considered small, we define a *maximum percent error* $\epsilon_{i,\,\max}(\xi, p)[\%]$ $(i = 1, 2)$ as

$$\epsilon_{i,\,\max}(\xi, p)[\%] = \frac{\Theta_i(\xi, \tau = 0) - \Psi_i(\xi, \tau = 0, p)}{\Theta_i(\xi, \tau = 0)} \times 100$$

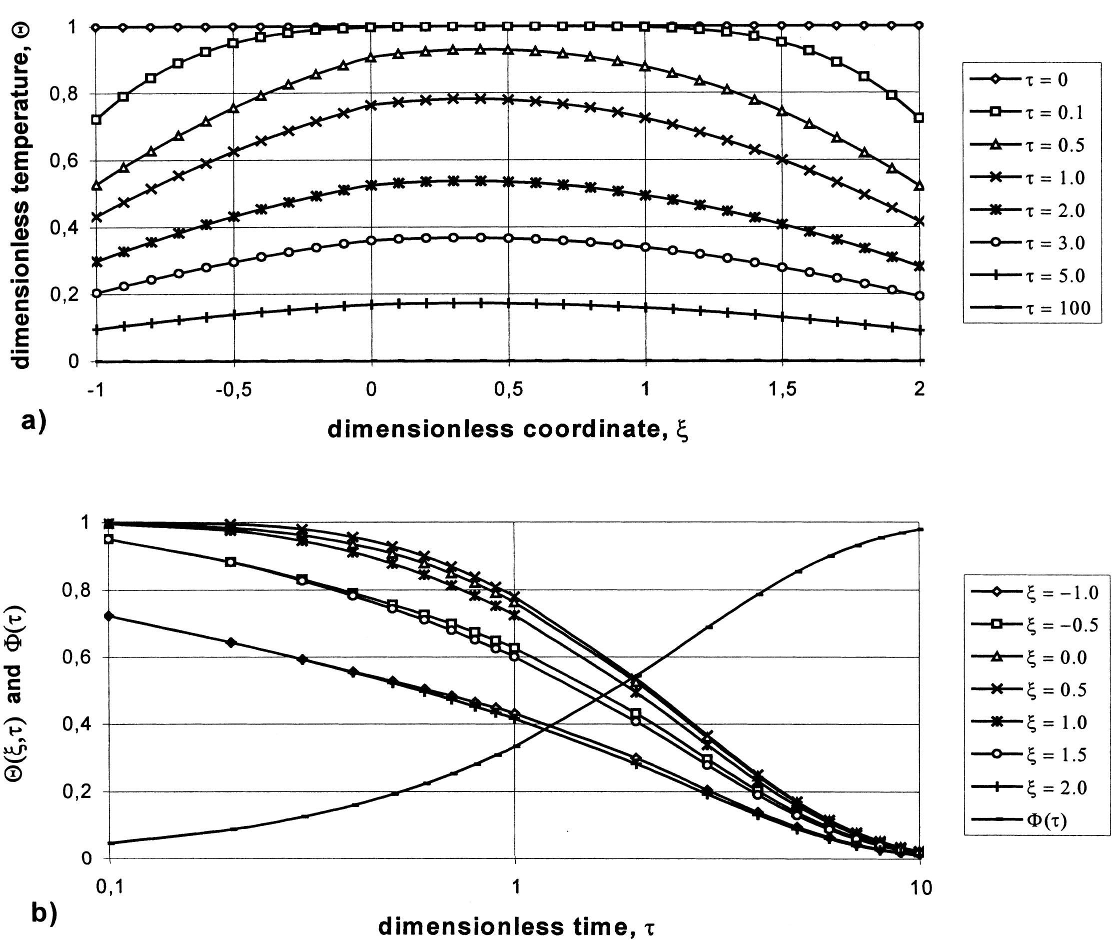

Fig. 4. Transient-temperature chart for a two-layer slab having $\gamma = 2$, $\kappa = 2$, $\delta = 1$, $Bi_1 = 1$ and $Bi_2 = 2$. (a) Dimensionless temperature vs. ξ with τ as a parameter. (b) Dimensionless temperature and heat transferred vs. τ (with ξ as a parameter for only Θ).

where $\Theta_i(\xi, \tau = 0) = 1$. Graphs of $\epsilon_{i,\max}(\xi, p)$ are plotted in Fig. 3(b). When $p = 9$ the maximum percent error is maximum at $\xi = \gamma = 2$, where it is equal to about 7%. Instead, when $p = 17$, the maximum percent error is still maximum at $\xi = \gamma = 2$, where it is equal to about 3.6%.

Therefore, when $p = 17$, the exact and approximate dimensionless temperatures differ by not more than 3.6% in the heaviest conditions, i.e. for $\tau = 0$ and $\xi = \gamma = 2$. Although for sufficiently large time τ the number p can notably be reduced, we fix $p = 17$ for the calculation of temperature and heat exchanged for both small and large times in view of the availability today of a high computer technology.

5.3. Dimensionless temperature and heat transferred

Once the number p of eigenvalues has been established, the dimensionless temperatures $\Theta_1(\xi, \tau)$ and $\Theta_2(\xi, \tau)$, as well as the dimensionless heat exchanged $\Phi(\tau)$, may be calculated. The results are graphically presented in Fig. 4.

Fig. 4(a) gives the dimensionless temperature Θ as a function of the dimensionless position ξ for several different values of the parameter τ. For $\tau = 0$, we have $\Theta = 1$. For large values of τ, the dimensionless temperature becomes small according to the thermal equilibrium (steady end-state) which is going to be reached between slab and surrounding fluid.

Fig. 4(b) relates the temperature Θ to different times τ with ξ as a parameter. Fig. 4(b) also shows the dimensionless heat Φ transferred during the transient process as a function of the dimensionless time τ. Once ξ and τ have been given, Fig. 4(b) allows Θ and Φ for the considered two-layer slab to be graphically evaluated.

6. Conclusions

A 'natural' analytic approach for solving one-dimensional transient heat conduction in a composite slab, whose layers are in perfect thermal contact, has been developed. It combines the efficiency of Tittle's approach for the determination of the eigenvalues with the simplicity of Vodicka's approach for the calculation of the orthogonal eigenfunctions. The combined method is relatively simple and particularly convenient when compared with classical methods heretofore employed.

The obtained solution gives a complete description of the thermal field in any one of the two layers and allows the heat exchanged between composite slab and surrounding fluid to be established. The exact closed-form solution is of the form of an infinite series in the space-variable with an exponential time dependency where the thermal diffusivity of the first layer explicitly appears. Dimensionless variables and groups have been introduced to simplify the representation of the formal solution to the transient two-layer problem.

A numerical example has allowed the slab temperature profile and the heat exchanged with the surrounding fluid during the transient process to be calculated. The results have been presented in the form of charts for ready reference. No special equation for very small time has been derived, as well as no particular approximation for sufficiently large time has been used, in view of the very high computing technology available today.

The method of analysis can be applied to composites of any number of layers, although solutions for only two material composite slabs have been presented in this paper.

Appendix A

The functions $\Pi_i(\lambda_i)$ $(i = 1, 2)$, appearing in Eqs. (12) and (13), are:

$$\Pi_i(\lambda_i) = \frac{k_i\lambda_i + h_i \tan(\lambda_i L_i)}{h_i - k_i\lambda_i \tan(\lambda_i L_i)}$$

Bearing in mind Eq. (17) and setting $\lambda_1 = \lambda$, the functions Π_i become

$$\Pi_1(\lambda) = \frac{k_1\lambda + h_1 \tan(\lambda L_1)}{h_1 - k_1\lambda \tan(\lambda L_1)}$$

$$\Pi_2(\lambda) = \pm\Pi_2'(\lambda)$$

$$\Pi_2'(\lambda) = \frac{k_2\sqrt{\alpha_1/\alpha_2}\lambda + h_2 \tan\left(\sqrt{\alpha_1/\alpha_2}\lambda L_2\right)}{h_2 - k_2\sqrt{\alpha_1/\alpha_2}\lambda \tan\left(\sqrt{\alpha_1/\alpha_2}\lambda L_2\right)}$$

where the positive sign has to be chosen when the separation constant ratio, λ_2/λ_1, defined by Eq. (17) is positive; conversely, the negative sign when this ratio is negative. When normalized variables are used, the functions $\Pi_1(\lambda)$ and $\Pi_2'(\lambda)$ become

$$\Pi_1(\beta) = \frac{\beta + Bi_1 \tan(\beta)}{Bi_1 - \beta \tan(\beta)}$$

$$\Pi_2'(\beta) = \frac{\left[\left(\gamma/\sqrt{\delta}\right)\beta\right] + (Bi_2\gamma/\kappa) \tan\left[\left(\gamma/\sqrt{\delta}\right)\beta\right]}{(Bi_2\gamma/\kappa) - \left[\left(\gamma/\sqrt{\delta}\right)\beta\right] \tan\left[\left(\gamma/\sqrt{\delta}\right)\beta\right]}$$

A new function called $\Pi_2''(\beta)$ has been introduced in Eq. (46). It is defined as

$$\Pi_2''(\beta) = -\frac{\sqrt{\delta}}{\kappa}\Pi_2'(\beta).$$

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